1. Gradients & Directional Derivatives

First, it is important to recall how we have been writing partial derivatives. For example, we can represent the partial derivatives of $f(x, y)$ as

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}. \tag{1.1}$$

Or more generally for a function with $x_1, x_2, \ldots, x_n$ as its parameters,

$$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}. \tag{1.2}$$

We can capture this collection of partial derivatives in something called the *gradient.*

Definition 1.1 (Gradient)**.** For any function $f(x_1, x_2, \ldots, x_n)$, we can write the gradient as

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n} \rangle. \tag{1.3}$$

Example 1.1 (Gradient)**.** Let $f(x, y) = x^2 + y^2$. We can write the gradient as

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 2x, 2y \rangle. \tag{1.4}$$

Now if we want to find the gradient at a specific point, say $(x, y) = (3, 2)$, we can simply plug x, y into $\nabla f(x, y)$ which becomes $\nabla f(3, 2) = \langle 6, 4 \rangle$.

Remark 1.1. We are used to taking derivatives in a specific "direction". If we consider the standard basis vectors for $\mathbb{R}^2$, taking a derivative with respect to x results in the rate of change in the $\hat{\imath}$ direction. What if we want to get the rate of change or derivative in any arbitrary direction?

Definition 1.2 (Directional derivative)**.** The directional derivative of f,

$$D_{\mathbf{v}} f = \nabla f \cdot \mathbf{v} \tag{1.5}$$

is the rate the function f change when we move in the direction of $\mathbf{v}$, $\hat{v}$. To understand this geometrically, we know that the dot product of $\mathbf{a}$ onto $\mathbf{b}$ is "how much of" $\mathbf{a}$ is onto $\mathbf{b}$. So by storing all partial derivatives of a function in a vector, we can obtain a scalar value of how much a function is changing in a specific direction.

Example 1.2 (Directional derivative)**.** We want to know how much the function $f(x, y) = x^2 + y^2$ is changing at $(x, y) = (3, 2)$ along the direction $\hat{v} = \langle 0.96, -0.28 \rangle$. First, we must find the gradient of f at $(3, 2)$ via

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y \rangle \tag{1.6}$$

$$\nabla f(3, 2) = \langle 2(3), 2(2) \rangle = \langle 6, 4 \rangle. \tag{1.7}$$

Then, by the previous definition,

$$D_{\hat{v}} f = \nabla f(3, 2) \cdot \langle 0.96, -0.28 \rangle \tag{1.8}$$

$$= \langle 6, 4 \rangle \cdot \langle 0.96, -0.28 \rangle \tag{1.9}$$

$$= 4.64 \tag{1.10}$$

is our derivative in the direction of $\hat{v}$ at $(x, y) = (3, 2)$.

2. Tangent Plane?

Consider the equation $z = x^2 + y^2$. Think of this as a countour (level surface)

$$G(x, y, z) = 0, \quad x^2 + y^2 - z = 0. \tag{2.1}$$

A tangent plane means no change in $G(x, y, z)$ which implies $\mathrm{D}_{\mathbf{u}}G = 0$ for all directions $\mathbf{u}$.

$$\nabla G \cdot \mathbf{u} = 0 \tag{2.2}$$

$$\nabla G = 0 \tag{2.3}$$

Definition 2.1 (Tangent plane)**.** Consider the function $f(x, y, z)$ where the tangent plane at the point (x_1, y_1, z_1) is

$$\frac{\partial f}{\partial x}(x - x_1) + \frac{\partial f}{\partial y}(y - y_2) + \frac{\partial f}{\partial z}(z - z_0). \tag{2.4}$$