[MA113] Unofficial Practice Final

This is an unofficial practice final exam for Calculus III containing some problems I found pretty tough. These problems are not designed to be easy; you'll most likely be fine if you can do them. Please note that these are not in any particular order of difficulty.

Compiled for the Fall quarter of the year 2025.

Contents

1. Vectors and Vector-valued Functions

Problem 1.1. The following is an easier problem, just to get warmed up. Let $\mathbf{v} = \langle 3, -5, 7 \rangle$ and $\mathbf{w} = \langle 4, 1, -8 \rangle$. First, compute $\mathbf{v} \cdot \mathbf{w}$, and $\mathbf{v} \times \mathbf{w}$ and then find $\text{proj}_{\mathbf{w}}(\mathbf{v})$.

Problem 1.2. A projectile is attached to a string and spinning such that its position is $\langle \cos(t^2), 5 + \sin(t^2) \rangle$ meters. When the speed of the object reached 10 meters per second, the string breaks and the projectile is flung off tangent to the circle described by the string-constrained motion. The projectile does not experience air resistance and so travels on a parabolic path. The ground is a hill through (0,0) making an angle of 30° downwards. Where, in vector form, does the projectile land?

Problem 1.3. When an object is at position $(3, 9, -2)$, its velocity is $\langle -1, -4, -8 \rangle$ and its acceleration is $\langle -5, 4, 3 \rangle$. **(a)** Determine the unit tangent vector. **(b)** Determine the tangential and normal acceleration vectors. **(c)** Determine the parametric equations for the equation of the osculating circle.

2. Partial Derivatives

Problem 2.1. Prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^4+y^2}$ does not exist.

Problem 2.2. When Isaak minimizes $f(x,y)$ the critical points are found by setting the gradient $\nabla f(x,y) = \langle 4xy, 4x^2 - 4y\rangle$ to the zero vector, $\mathbf{0}$. Explain why we know that Isaak did not calculate the gradient correctly.

Problem 2.3. The surface $z = f(x,y)$ satisfies $f(3,6) = 17$ and $\nabla f(3,6) = \langle 6, -5\rangle$. **(a)** Find the equation of the plane tangent to the surface at $(3,6,17)$. **(b)** Use a linearization to estimate $f(3.1, 5.7)$. Show your work.

Problem 2.4. $f(x, y)$ is a function with gradient

$$\nabla f(x, y) = \langle 4x^2 - y^2, -2xy - 4y - 3 \rangle. \tag{2.1}$$

Find the critical points and use the second derivative test to classify each critical point.

Problem 2.5. Find the points on the curve $(x^2 + y^2)^3 = (x^2 - y^2)^2$ that are closest to and farthest from the point $(2, -4)$.

3. Double and Triple Integrals

Problem 3.1. Find the volume of the region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$.

Problem 3.2. What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2)\mathrm{d}V. \tag{3.1}$$

Please give a reason for your answer (show work).

Problem 3.3. Suppose $f(x,y)$ is continuous over a region R in the plane and that the area $A(R)$ of the region is defined. If there are constants m and M such that $m \leq f(x,y) \leq M$ for all $(x,y) \in R$, prove that

$$mA(R) = \iint_R f(x,y)\mathrm{d}A \leq MA(R). \tag{3.2}$$

Problem 3.4 A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

Problem 3.5. Show that planes perpendicular to the x-axis have equations in the form of $r = a\sec(\theta)$ in cylindrical coordinates and that planes perpendicular to the y-axis have equations of the form $r = b\csc(\theta)$. Then, find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c$ where $c \neq 0$.

Problem 3.6. I don't think this is very hard, so here is a little break. Set up triple integrals for the volume of the sphere $\rho = 2$ **(a)** cylindrical, **(b)** spherical, and **(c)** cartesian coordinates.

Problem 3.7. Consider the integral,

$$\int_0^5 \int_{x^2}^{5x} (x + \sqrt{y}) \mathrm{d}x. \tag{3.3}$$

(a) Switch the order of integration. **(b)** Evaluate both the original and the switched integral and comment intelligently on your results.

Problem 3.8. Find the volume of the solid region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

Problem 3.9. Find the average value of the function $f(\rho, \phi, \theta) = \rho \cos(\theta)$ over the upper half of the solid ball $\rho < 1, 0 \leq \phi \leq \frac{\pi}{2}$.

Problem 3.10. Find the average value of the function $f(r, \theta, z) = r$ over the solid region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.

Problem 3.11. Find the centroid (center of mass) of the solid region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.

Problem 3.12. Find the Jacobian of the transformation $x = u$, $x = uv$ and sketch the region $\mathbf{G} : 1 \leq u \leq 2, 1 \leq uv \leq 2$, in the wv-plane.

Problem 3.13. A thin plate of constant density covers the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate this integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk $\mathbf{G} : u^2 + v^2 \geq 1$ in the uv-plane.

Problem 3.14. A basin lies inside the circle $x^2 + y^2 = 4\text{m}$ with height

$$z = \frac{(x^2 + y^2)^2 - (x^2 + y^2) - 12}{10}\text{m}. \tag{3.4}$$

Overnight, rain falls over the area. If the ground had been flat, the depth of the rainfall would have been

4. Miscellaneous Topics

Problem 4.1. Use Taylor's formula to find a quadratic approximation of $f(x, y) = \cos(x)\sin(y)$ at the origin. Estimate the error in the approximation if $|x| \leq 1$ and $|y| \leq 0.1$.

Problem 4.2. This is a very similar example as the one above, but more practice never hurts. Use Taylor's formula to find a quadratic approximation of $f(x, y) = e^x \sin(y)$ at the origin. Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

Problem 4.3. Find the length of the parabolic segment $r = \frac{2}{1-\cos(\theta)}$.

Problem 4.4. If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$, with respect to θ is given by the formula

$$r_{\text{avg}} = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} f(\theta) \mathrm{d}\theta. \tag{4.1}$$

Use this formula to find the average value of r with respect to θ over the following curves:
(a) The cardioid $r = a(1 - \cos(\theta))$. **(b)** The circle $r = a$. **(c)** The circle $r = a\cos(\theta)$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Make sure to show work.