# [MA221] DIFFERENTIAL EQUATIONS AND MATRIX ALGEBRA NOTES

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Last modified: March 6, 2026

These notes were written for [MA221] Differential Equations and Matrix Algebra I taken with Dr. Goulet during the Winter quarter of 2026. Use and distribute this as you feel but note that these were taken by me as notes and like all notes, I did not write down everything and the information I did write down might not be completely accurate.

## CONTENTS

# 1. First Order Partial Derivatives

Recall that the chain rule is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}g(u(x)) = g'(u(x))u'(x) = \frac{\mathrm{d}g}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{1.1}$$

From Calculus II, we also know the *substitution method*

$$g(u(x)) = \int g'(u(x))u'(x)\mathrm{d}x = \int g'(u)\mathrm{d}u. \tag{1.2}$$

**Example 1.1.** Say we have some implicit solution $a(y) = b(x)$. First, we should take the derivative of both to see which differential equation our solution would solve:

$$a'(y)y'(x) = b'(x). \tag{1.3}$$

So then

$$y'(x) = \frac{b'(x)}{a'(y)} = b'(x)\frac{1}{a'(y)}. \tag{1.4}$$

**Definition 1.1** (Separable ODE)**.** An ODE in the form $y' = f(x)g(y)$. We can separate the ordinary differential equation by doing

$$\frac{1}{g(y)}y' - f(x) \tag{1.5}$$

$$\int \frac{1}{g(y(x))}y'(x)\mathrm{d}x = \int f(x)\mathrm{d}x \tag{1.6}$$

which gives us our final solution

$$\frac{1}{g(y)}\mathrm{d}y = \int f(x)\mathrm{d}x. \tag{1.7}$$

**Example 1.2** (Separable ODE)**.** An exmaple would be

$$y' = \frac{y}{1+x} \tag{1.8}$$

$$= \frac{1}{1+x}y \tag{1.9}$$

where $f(x) = \frac{1}{1+x}$ and $g(y) = y$.

**Example 1.3** (Separable ODE)**.** Another good example is $(e^{2y} - y)\cos(x)\frac{\mathrm{d}y}{\mathrm{d}x} = e^y \sin(2x)$. We can take its derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{e^y}{e^{2y} - y}\right)\left(\frac{\sin(2x)}{\cos(x)}\right) \tag{1.10}$$

which does prove that this equation is indeed separable.

**Example 1.4.** Another good example is the *logistic model of population growth* defined via

$$P' = k(1 - \frac{P}{L})P. \tag{1.11}$$

First, we can write this as

$$\frac{1}{(1 - \frac{P}{L})P} P' = k \tag{1.12}$$

and then proceed to apply the chain rule and integrate

$$\int \frac{1}{(1 - \frac{P(t)}{L})P(t)} P'(t)\mathrm{d}t = \int k\mathrm{d}t \tag{1.13}$$

which then means that

$$\int \frac{1}{(1 - \frac{P}{L})P} \mathrm{d}P = kt + C. \tag{1.14}$$

Now, we can use *partial fraction decomposition* and solve for the values of $A$ and $B$:

$$\frac{1}{(1 - \frac{P}{L})P} = \frac{A}{1 - \frac{P}{L}} + \frac{B}{P} = \frac{AP + B(1 - \frac{P}{L})}{(1 - \frac{P}{L})P} \tag{1.15}$$

$$\therefore B = 1, A = \frac{1}{L}. \tag{1.16}$$

Blah blah blah then you can keep going and try to solve for an implicit solution.

*Remark* 1.1. Unless otherwise specified, make sure to find the explicit solution.

**Notes for 12/5/2025.**

*Remark* 1.2. Can we make a plot of the solution without solving the ODE?

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy$ and $y(1) = 2$. In theory, we could try to create a graph of the slopes (*slope field*) to try to approximate where the solutions would end up by picking some point on the slope field.

**Definition 1.2** (Autonomous ODE)**.** An ODE in which the independent variable (usually $x$) does not appear explicitly. For example, $y' = 1 + y^2$ would be considered autonomous even if we can rewrite this as $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$.

These types of ODE are quite common in application since the independent variable is often time. This would make sense since our universal and physical laws governing the world are usually time independent.

**Example 1.5.** Consider the separable ODE $y' = 1 + y^2$ where $y(1) = 0$. First, we should separate the equation and integrate:

$$\int \frac{1}{1+y^2}\mathrm{d}y = \int 1\mathrm{d}x \tag{1.17}$$

which evaluates to

$$\arctan(y) = x + C \implies y = \tan(x + C). \tag{1.18}$$

Now, if we substitute our original data into our equation, we see that $C = k\pi$ where $k \in \mathbb{Z}$. This means that as of now, $y = \tan(x + k\pi)$. However, we also recognize that the period of $\tan(x)$ is $\pi$, meaning we can just present our final solution as $\boxed{y = \tan(x)}$.

*Remark* 1.3. For autonomous ODEs, we can create a *phase diagram*. This is a quick diagram that uses the first derivative test to *simply* describe the behavior of our ODE's solution.

# 2. Existence and Uniqueness of Solutions

Consider the differential equation $y' = -\sqrt{y}$. If we find the explicit solution, we see that

$$y = \left(\frac{t+C}{2}\right)^2. \tag{2.1}$$

Now, looking back at the original equation, we see that $y = 0$ satisfies the equation despite the fact that there is no constant $C \in \mathbb{R}$ that gives us $y = 0$. The reason for this is that we need to specify a certain range or set of numbers that are supported by our equation. Technically, in solving the differential equation, we needed to divide by $y$ which means dividing by 0. And since when the water level is 0 it cannot decrease more, we need value of $y > 0$.

If we consider some initial data like $y(0) = 1$, we can see that $C = \pm 2$. We can then see that we have two solutions to our equation defined via

$$y = \left(\frac{t+2}{2}\right)^2, \quad y = \left(\frac{t-2}{2}\right)^2. \tag{2.2}$$

Are these valid solutions? Well, there are different solutions for different intervals. For example, $y = \left(\frac{t+2}{2}\right)^2$ only holds for $t \leq -2$ which cannot possibly be a solution because we have initial data $y(0) = 1$. However, our other solution, $y = \left(\frac{t-2}{2}\right)^2$, is defined for all $t \leq 2$ which does include $t = 0$. Therefore, that is a valid solution to our differential equation.

**Theorem 2.1** (PicardLindelf Theorem)**.** *If $y'(x) = f(x, y(x)), y(x_0) = y_0$ and there exists a set of points $a \leq x \leq b$, $c \leq y \leq d$ with $(x_0, y_0)$ inside the set and if the function $f$ is continuous inside the set, then there is a solution $y(x)$, valid in some interval $x_0 - h \leq x \leq x_0 + h$. If, in addition, $\frac{\partial f}{\partial y}$ is continuous in this set, then the solution is unique in that interval.*

*Remark* 2.1. Existing and unique solutions are the only ones that really matter in some physical capacity. The others are basically useless to us.

**Practice Problems for 12/12/2025.**

**Problem 1.** Let $G(t)$ be the amonut of gold (in mL) of gold in the tank and let $V(t)$ be the total volume of fluid in the tank. We can set up the two differential equations

$$\frac{\mathrm{d}G}{\mathrm{d}t} = R_{\text{in}}C_{\text{in}} - R_{\text{out}}C_{\text{out}} = 10\frac{1}{2} - 2\frac{6}{V} \tag{2.3}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = R_{\text{in}} - R_{\text{out}} = 10 - 2 = 8. \tag{2.4}$$

We know that $V' = 8$ and by integrating, we get that

$$V(t) = 8t + C. \tag{2.5}$$

Now, we need to solve for the integration constant $C$ by plugging in $V(0) = 100$ which gives us $C = 100$. We can the rewrite $V$ as

$$V(t) = 8t + 100. \tag{2.6}$$

Now let us go back and consider the differential equation

$$G' = 5 - 2\frac{G}{8t + 100}, \qquad G(0) = 100. \tag{2.7}$$

Since this equation is linear but not separable, we can use the *variation of parameters* technique to solve it. First, we can rewrite it as

$$G' + \frac{1}{4t + 50}G = 5 \tag{2.8}$$

$$\frac{1}{G}G' = -\frac{1}{4t + 50}. \tag{2.9}$$

Okay honestly, I completely forgot about this so you don't get to have this problem completed. Oopsies, my bad!

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dsolve([P'(t) = a * P(t) - b * P(t)^2, P(0) = 400])
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# 3. NUMERICAL METHODS OF DIFFERENTIAL EQUATIONS

Say we plan to solve $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y(t), t)$. From Calculus I, we can say that

$$y'(t_0) \approx \frac{y(t_1) - y(t_0)}{t_1 - t_0}. \tag{3.1}$$

With a starting $x, y$ value, with the equation

$$y(t_1) \approx y(t_0) + \Delta t f(y(t_0), t_0) \tag{3.2}$$

we can approximate the next values of $y$. As you may recall, this is known as *Euler's method.* In theory, we can continue this approximation as much as we want but unfortunately there is error that does accumulate.

**Definition 3.1** (Forward Euler's Method with Fixed Step Size)**.** As aforementioned, Euler's method is defined via Equation 3.2. However, the idea is that we are building an approximation from the previous approximation plus some step size and some error. A quick and not very rigorous derivation of Euler's method is as follows:

$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(y(s), s)\mathrm{d}s \approx \Delta t \cdot f(y(t_0), t_0). \tag{3.3}$$

However, this is a poor approximation as we treat the integral of $f(y(t), t)$ as a rectangle, which it is not. Luckily, we can get a better approximation by taking the area as a trapezoid instead of a rectangle which yeilds a more accurate approximation of an integral. We can write our new approximation as

$$y(t_1) \approx y(t_0) + \frac{1}{2}k\left[f(y(t_0), t_0) + f(y(t_1), t_1)\right]. \tag{3.4}$$

This is a good approach, but there is a massive issue. We do not know the value of $y(t_1)$, that is the value we are *trying* to approximate anyway. We must call upon a different method built off of Euler's method, *Heun's Method.*

**Definition 3.2** (Heun's Method)**.**

$$y_{\text{temp}} = y_n + kf(y_n, t_n) \tag{3.5}$$

$$y^{n+1} = y_n + \frac{1}{2}k\left[f(y_n, t_n) + f(y_{\text{temp}}, t_{n+1})\right]. \tag{3.6}$$

This is a more "accurate" method of Euler's method that uses trapezoid for approximations instead of rectangles. We simply plug in our values and blah blah blah. Actually, there isa more natural way of doing this computationally:

$$k_1 = hf(y_n, x_n) \tag{3.7}$$

$$k_2 = hf(y_n + k_1, x_{n+1}) \tag{3.8}$$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2). \tag{3.9}$$

*Remark* 3.1. Other than more computation, there is no noticeable disadvantage of Heun's Method compared to Euler's method.

**Example 3.1** (Numerical Approximations)**.** Let $y' = y - 2x$ and $y(1) = 2$. If our solution to this differential equation was

$$y(x) = 2x + 2 - 2e^{x-1}. \tag{3.10}$$

For the purposes of this problem, even though we have $y(x)$, we will estimate $y(2)$ using both *Euler's method* and *Heun's Method*. We can rewrite Equation 3.10 as

$$y' = y - 2x = f(y_1, x) \tag{3.11}$$

which lets us perform Euler's method as follows:

$$\begin{aligned} y_1 &= y_0 + hf(y_0, x_0) && (3.12)\\ &= 2 + \frac{1}{2}(2 - 2 \cdot 1) && (3.13)\\ &= 2. && (3.14) \end{aligned}$$

Now, to find our next approximation, $y_2$, we need to find $x_1 = x_0 + h = 1 + \frac{1}{2} = \frac{3}{2}$. Then, we can say that

$$\begin{aligned} y_2 &= y_1 + hf(y_1 x_1) && (3.15)\\ &= 2 + \frac{1}{2}\left(2 - 2 \cdot \frac{3}{2}\right) && (3.16)\\ &= \frac{3}{2}. && (3.17) \end{aligned}$$

This can keep going for as long as it needs to. However, we can now perform the same numerical approximation using Heun's method. Let $x_0 = 1, y_0 = 2, h = \frac{1}{2}$ and

$$y_{\text{temp}} = y_0 + hf(y_0, x_0) = 2. \tag{3.18}$$

Now, we can estimate $y_1$ via

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}h\left[f(y_0, x_0) + f(y_{\text{temp}}, x_1)\right] && (3.19)\\ &= 2 + \frac{1}{2}\left(\frac{1}{2}\right)\left[0 + 2 - 2 \cdot \frac{3}{2}\right] && (3.20)\\ &= \frac{7}{4}. && (3.21) \end{aligned}$$

Similarly, we can numerically approximate $y_2$ by first finding

$$y_{\text{temp}} = y_1 + hf(y_1, x_1) = \frac{9}{8} \tag{3.22}$$

and then solving

$$y_2 = y_1 + \frac{1}{2}h\left[f(y_1, x_1) + f(y_{\text{temp}}, x_2)\right] = \frac{33}{32}. \tag{3.23}$$

We see that Heun's method is more accurate than Euler's method despite being more computationally troublesome to solve.

Now it would only make sense to discuss a "family" of numerical approximation methods. More specifically, the *Runge-Kutta methods.* Both Euler's and Heun's method fall into this category of iterative approximations. In the same way we developed upon Euler's method and discussed Heun's method, we can now discuss the *explicit midpoint method* which does indeed behave similarly to Heun's method.

**Definition 3.3** (Explicit Midpoint Method)**.** We can define the use of this method as

$$y_{\text{est}} = y_n + \frac{1}{2}hf(y_n, t_n) \tag{3.24}$$

$$y_{n+1} = y_n + hf\left(y_{\text{est}}, t_n + \frac{1}{2}h\right). \tag{3.25}$$

Actually, we can generalize this to an *explicit two-stage Runge-Kutta method* in the form

$$k_1 = hf(y_n, t_n), \qquad k_2 = hf(y_n + ak_1 + n + ah), \qquad y_{n+1} = y_n + bk_1 + ck_2. \tag{3.26}$$

**Example 3.2** (Runga-Kutta 4)**.** Let

$$k_1 = hf(y_n, t_n), \tag{3.27}$$

$$k_2 = hf\left(y_n + \frac{1}{2}k_1, t_n + \frac{1}{2}h\right), \tag{3.28}$$

$$k_3 = hf\left(y_n + \frac{1}{2}k_2, t_n + \frac{1}{2}h\right), \tag{3.29}$$

$$k_3 = hf\left(y_n + k_3, t_n + k\right). \tag{3.30}$$

Now, we can compute $y_{n+1}$ via

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4 \tag{3.31}$$

*Remark* 3.2*.* Some people consider Runga-Kutta 4 (or RK4) to be *the* Runga-Kutta method due to its relative simplicity and "easy-to-teach" nature. However, this is inaccurate as there is no singular RK method.

# 4. COMPLEX NUMBERS

Alas, it is time to discuss *complex numbers*. For instance, the equation $z^2 = -1$ has two solutions, $\pm i$. This solution is comprised of complex numbers.

**Definition 4.1** (Complex Number)**.** A complex number $k \in \mathbb{C}$ is written in the *Cartesian form* of $a + ib$ where $i^2 = -1$ and $a, b \in \mathbb{R}$. Complex numbers are equipped with addition and multiplication. Addition is defined via

$$(a + bi) + (c + di) \tag{4.1}$$

and multiplication of complex numbers is defined via

$$(a + bi)(c + di). \tag{4.2}$$

When performing division on complex numbers, we need to multiply both the numerator and denominator by the *complex conjugate* denoted via $\bar{z}$ for some $z \in \mathbb{C}$. The complex conjugate also has the property where $z\bar{z} = |z^2|$ and $|c + di| = \sqrt{c^2 + d^2}$ where $|z|$ is the *modulus* of $z$, the distance from $z$ to 0 in the complex plane.

Since we can think of complex numbers as 2-D numbers and can plot them, we can represent them as vectors. In the same way that the modulus is similar to the magnitude of the vector, the *principle argument* of $z \in \mathbb{C}$ is written as $\operatorname{Arg}(z)$, the standard polar angle in radians. However, if we now let $r = |z|$ and $\theta = \operatorname{Arg}(z)$, we can rewrite $z$ as

$$\begin{aligned} z = x + iy &= r\cos(\theta) + ir\sin(\theta) && (4.3) \\ &= r(\cos(\theta) + i\sin(\theta)) && (4.4) \end{aligned}$$

since in polar coordinates, $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Using this, we can actually derive the famous formula, *Euler's formula*,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta). \tag{4.5}$$

This is often seen as the famous equation $e^{i\pi} + 1 = 0$. We can also rewrite the complex number $z$ in *Euler's form* as $z = re^{i\theta}$. Now if we have some $w = qe^{i\phi}$, we can simply divide them via

$$\frac{z}{w} = \frac{re^{i\theta}}{qe^{i\theta}} = \frac{r}{q}e^{i(\theta - \phi)}. \tag{4.6}$$

*Remark* 4.1. One interesting observation is that by taking a look at the equations above, the multiplication of complex numbers actually results in rotation.

Of course, we can also raise complex numbers to powers of other numbers.

**Example 4.1.** To compute $(2 + 3i)^5$, we can of course multiply and distribute this out normally, but in polar form, we can simply write the solution as $r^5 e^{i5\theta}$.

# 5. SOLVING SYSTEMS OF EQUATIONS USING MATRICES

**Definition 5.1** (Linear Independence)**.** For some vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ and some scalars $c_1, c_2, \ldots, c_n$, the sum

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n \tag{5.1}$$

is a linear combination of vectors. $\{\boldsymbol{v}_i\}_{i=1}^n$ is called a linearly independent set of vectors if $c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n = 0$ only if $c_1 = c_2 = \cdots = c_n = 0$.

If a set is *dependent*, at least one $a_i = 0$ and

$$\boldsymbol{v}_i = -\left(\frac{a_1}{a_i}\boldsymbol{v}_1 + \frac{a_2}{a_i}\boldsymbol{v}_2 + \cdots + \frac{a_n}{a_i}\boldsymbol{v}_n\right). \tag{5.2}$$

So, $\boldsymbol{v_i}$ can be written as a linear combination of the other vectors.

**Definition 5.2** (Rank)**.** The rank of a set of vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ is the number of vectors left in that set after removing all linearly dependent vectors, leaving only independent vectors.

We can also define the *row rank* of some matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ to be the number of linearly independent rows. We can also define the column rank to be basically the same thing, so we just use one function $\operatorname{rank}(A)$. In short, the rank of a matrix is the number of pivots it has after being row-reduced to REF or RREF.

**Proposition 5.1.** *Row operations do not change the rank of a matrix.*

**Proposition 5.2.** $A\boldsymbol{x} = \boldsymbol{b}$ *is consistent iff.* $\operatorname{rank}(A) = \operatorname{rank}(A|\boldsymbol{b})$ *for some* $A \in \mathcal{M}_{m\times n}(\mathbb{R})$.

**Proposition 5.3.** *If* $A\boldsymbol{x} = \boldsymbol{b}$ *is a consistent system,* $n - \operatorname{rank}(A)$ *is the dimension of the solution space for some matrix* $A \in \mathcal{M}_{m\times n}(\mathbb{R})$.

If we have the equation $5x = 2$, we can solve for $x$ simply by multiplying both sides by the inverse of the real number 5, $5^{-1}$ or more commonly written as $\frac{1}{5}$. In the same way real numbers have inverses, that we can multiply to obtain the multiplicative identity, 1, matrices can also have *inverses.* However, this is not always guaranteed.

**Definition 5.3** (Inverse of a Matrix)**.** A matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ is considered to be invertible if it meets the following requirements

1. The matrix $A$ is square, in this case, $m = n$.
2. The matrix $A \sim I$. Meaning, $\operatorname{RREF}(A) = I$.

As we know, we can store row operations into matrices, in this case, lets say for $k$ row operations, we have real matrices $R_i$ that represent different row operations on $A$,

$$R_k \ldots R_3R_2R_1A = I \tag{5.3}$$

and let $R_k \ldots R_3R_2R_1 = B$, we can see that $BA = I$. Meaning, $B = A^{-1}$. So finally, we can write the inverse of the matrix $A$ as $A^{-1}A = I$.

**Proposition 5.4.** *If* $A^{-1}$ *exists, then* $A^{-1}A = I = AA^{-1}$.

**Definition 5.4** (Non-singular)**.** If $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $A^{-1}$ exists, then we say the matrix $A$ is non-singular. Note, saying a matrix is singular implies that $A^{-1}$ does not exist.

*Remark* 5.1. Simply because there exist two matrices $A, B$ such that $AB = I$, that does not mean those two matrices are invertible. For example,

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \\ 3/2 & -1/2 \end{pmatrix} \tag{5.4}$$

is an example where $AB = I$ but since they are not square matrices, they are not *true* inverses.

Often times, taking the inverse of a matrix is pretty useless and time consuming. However, the inverse of a $2 \times 2$ matrix in the form of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.5}$$

ends up being

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{5.6}$$

This formula ends up being quite useful if you have some simple system that needs solving where the entries of the matrix $A$ has unknown parameters as row operations involving variables can be quite tricky.

As we probably already know, for some $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, $A^{-1}$ exists if and only if

$$\det(A) = ad - bc = 0. \tag{5.7}$$

We can now further discuss the idea of *determinants*.

**Definition 5.5** (Determinant)**.** For a $2 \times 2$ real matrix, we can define the determinant, also commonly denoted by $|A|$, as

$$\det(A) = ad - bc. \tag{5.8}$$

Now, for some arbitrary matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, if we pick some $(A)_{ij}$, remove the row and column containing $(A)_{ij}$. The determinant of this smaller matrix is called a *minor determinant*, $(M)_{ij}$. We can also define the the *cofactor*,

$$(C)_{ij} = (-1)^{i+j}(M)_{ij}, \tag{5.9}$$

which can be used to compute the value of $\det(A)$.

**Example 5.1.** Consider the following matrices that follow Definition 5.5:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}, \quad M_{32} = \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad C_{32} = (-1)^{3+2} \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = -1. \tag{5.10}$$

**Proposition 5.5.** *If we pick any row (or column) of some matrix $A$, compute all of the cofactors for every element in that row. Then,*

$$\det(A) = \sum_{j=1}^{n} (A)_{ij}(C)_{ij}. \tag{5.11}$$

*However, the visual definition is often easier to understand and apply.*

*Remark* 5.2. The concept of manually taking the determinant of any matrix larger than $3\times 3$ is absolutely disgusting, if you have an $n \times n$ matrix, it will require $\frac{n!}{2}$ calculations of $2 \times 2$ matrix determinants. However, there are some tricks to make their computation faster.

As you could probably guess from (5.6), the following proposition becomes quite obvious.

**Proposition 5.6.** *For some matrix $A$, $A^{-1}$ exists iff.* $\det(A) \neq 0$.

As mentioned earlier, there are some tricks we can use to make the computation of determinants much easier. This involves taking advantage o and exploiting the zeroes in a matrix. For example, if we have a matrix

$$M = \begin{pmatrix} 2 & 0 & a \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix}, \tag{5.12}$$

it is valid to ask for what value of $a$ is $M$ singular (has an inverse). Since we can see that column 2 has the most zeroes, we can let our cofactors be along that column.

$$\det(M) = 0 \cdot C_{12} + 0 \cdot C_{22} + 2 \cdot C_{32} \tag{5.13}$$

$$= (2)(-1) \det \begin{pmatrix} 2 & a \\ 1 & 1 \end{pmatrix} \tag{5.14}$$

$$= -2(2 - a). \tag{5.15}$$

So, $M$ is singular (no inverse) when $a = 2$ since $-2(2 - 2) = 0$ meaning $\det(A) = 0$.

We can also artificially create matrices with extra zeroes by row reducing a matrix into a *triangular matrix* and specifically a lower triangular matrix. However, remember the following when performing row operations on the determinant of a matrix:

1. A row swap will also swap the sign of $\det(A)$.
2. Row addition leaves $\det(A)$ unchanged.
3. Scalar multiplication multiplies $\det(A)$ by the same factor.

Although, often times it is easier to just work with the zeroes you already have instead of trying to create them via row operations.

# 6. Eigenvalues and Eigenvectors

Finally, we can discuss eigenvalues. We are used to solving systems in the form $A\boldsymbol{x} = \boldsymbol{b}$. However, if we have an equation in the form

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \tag{6.1}$$

where $\lambda \in \mathbb{R}$, we can rewrite it by subtracting $A\boldsymbol{x} - \lambda\boldsymbol{x} = 0$. Then, we can insert an identity matrix to allow us to factor out the vector $\boldsymbol{x}$ which gives us

$$A\boldsymbol{x} - \lambda I\boldsymbol{x} = (A - \lambda I)\boldsymbol{x} = 0. \tag{6.2}$$

Now if we let $M = A - \lambda I$, we can begin to solve the system $M\boldsymbol{x} = 0$. Since this system always allows $\boldsymbol{x} = \boldsymbol{0}$, we can try to get the *non-trivial* solutions which requires the existence of $M^{-1}$ which exists iff. $\det(M) \neq 0$. So, $A\boldsymbol{x} = \lambda I$ will have non-trivial solutions if and only if

$$\det(A - \lambda I) = 0. \tag{6.3}$$

So, we must choose a $\lambda$ to make $A\boldsymbol{x} = \lambda\boldsymbol{x}$ true. The value of $\lambda$ is called an *eigenvalue* and all the non-zero $\boldsymbol{x}$'s we find are called *eigenvectors.*

We can formalize the idea of an eigenvalue and eigenvector by forming a proper definition.

**Example 6.1.** Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}. \tag{6.4}$$

If we have the system

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \tag{6.5}$$

and begin by subtracting $\lambda I$ which yields

$$\left[ \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.6}$$

As we discussed earlier, there only exists a non-trivial solution iff.

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{pmatrix} = 0 \tag{6.7}$$

which we can easily compute via (5.5), resulting in

$$(1-\lambda)(2-\lambda) - 2 = 0 \implies \lambda = 0, 3. \tag{6.8}$$

We then substitute all values of $\lambda$ back into $(A - \lambda I)\boldsymbol{x} = 0$ to solve for the set of eigenvectors.

*Remark* 6.1. There are always *infinitely* many eigenvectors, if you find just one, you did something very wrong. Eigenvectors will usually be written as $c\boldsymbol{x}$ where $c \in \mathbb{R}$.

If we have the system $A\boldsymbol{x} = \lambda\boldsymbol{x}$ where $\lambda \in \mathbb{C}$, each value of $\lambda$ also admits a conjugate, $\overline{\lambda}$. Instead of redoing work to find the eigenvector of a conjugate of a complex number, we can simply take $\overline{A\boldsymbol{x}} = \overline{\lambda\boldsymbol{x}}$ which gives us $A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$. This idea leads up to the following proposition.

**Proposition 6.1.** *Following from the previous paragraph, for every $\boldsymbol{x}$ that is an eigenvector of $\lambda \in \mathbb{C}$, the vector $\overline{\boldsymbol{x}}$ is an eigenvector of $\overline{\lambda}$.*

**Example 6.2** (Markov chain)**.** If we consider some forest that is divided into areas where trees can grow. There are two types of trees that grow, red oaks and hickory.

At time $n$, the spot we are observing had a hickory tree, then the probability it still does at time $n+1$ is $\frac{3}{4}$. Now, if at time $n$ had a red oak, the probability it still does at time $n+1$ is $\frac{1}{2}$. Let the probability of there being hickory and red oaks at time $n$ be denoted via $h_n$ and $r_n$ respectively. We can write

$$h_{n+1} = \frac{3}{4}h_n + \frac{1}{2}r_n, \qquad r_{n+1} = \frac{1}{4}h_n + \frac{1}{2}r_n. \tag{6.9}$$

Now, let us describe this as a system of vector and matrices:

$$\begin{pmatrix} h \\ r \end{pmatrix}_{n+1} = \begin{pmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} h \\ r \end{pmatrix}_n . \tag{6.10}$$

Now, we question the existence of equilibrium and ask if $\lim\limits_{\boldsymbol{x}\to\infty} \boldsymbol{x}_n$ exists. We can observe the system $\boldsymbol{x}_{n+1} = A\boldsymbol{x}_n$ and let $\boldsymbol{x}_{\text{eq}} = A\boldsymbol{x}_{\text{eq}}$ and simply solve $A\boldsymbol{x} = \lambda\boldsymbol{x}$. By solivng for the eigenvalues, we can find the eigenvectors which end up telling us

$$\begin{pmatrix} h \\ r \end{pmatrix}_n = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} (1)^n + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \left(\frac{1}{4}\right)^n . \tag{6.11}$$

Then, as $n \to \infty$, we have

$$\lim_{n\to\infty} \begin{pmatrix} h \\ r \end{pmatrix}_n = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} . \tag{6.12}$$

If we end up going to equilibrium, we will have the vector $\begin{pmatrix} 2/3 & 1/3 \end{pmatrix}^\mathsf{T}$ which tells us that at equilibrium, $2/3$ of the trees will be hickory and $1/3$ of the trees will be red oak.

**Notes for 1/26/2026** (virtual)

If an eigenvalue has multiplicity $m$, then the dimension of the eigenspace could be $1, 2, 3, \ldots, m$.

**Definition 6.1** (Multiplicity)**.** Fill this in later.

**Definition 6.2** (Transpose of a matrix)**.** We define the transpose of a matrix, $A^\intercal$ where we basically just swap the rows and columns, not much more needs to be said about this.

Suppose we solved the system $A\boldsymbol{x} = \lambda\boldsymbol{x}$ in which we found $\lambda$ and $\boldsymbol{x}$. It turns out that we have partial solved $A^\intercal\boldsymbol{y} = \lambda\boldsymbol{y}$. Because we can then do something stupid like

$$(A^\intercal - \lambda I)\boldsymbol{y} = 0. \tag{6.13}$$

**Proposition 6.2.** *For some* $A \in \mathcal{M}_{m\times n}(\mathbb{R}), \det(A) = \det(A^\intercal)$.

**Proposition 6.3.** *$A$ and $A^\intercal$ share the same eigenvalues but not necessarily the same eigenvectors. Eigenvectors are the same if the matrix is diagonal or symmetric.*

**Proposition 6.4.** *If $A^{-1}$ exists, the eigenvalues are reciprocals of the eigenvalues of $A$ with the same eigenvectors. So, if we know the eigenvalues and eigenvectors of $A$ and there exists some $A^{-1}$, we know the eigenvectors and eigenvectors of $A^{-1}$.*

# 7. REVISITING DIFFERENTIAL EQUATIONS

Recall that the definition of a *linear ordinary differential equation* (ODE) is a differential equation written in the form

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = b(x) \tag{7.1}$$

with initial data

$$y(x_0) = y_0, y'(x_0) = y_0', \ldots, y^{n-1}(x_0) = y_0^{n-1}. \tag{7.2}$$

We can answer two fundamental questions regarding ODEs:

1. Does it have a solution (existence)?
2. If so, does it have more than one (uniqueness)?

**Theorem 7.1.** *If $a_i(x)$ are continuous in an interval around $x_0$ and if $a_n(x) \neq 0$, then there is one, and only one, solution to the equation.*

If we can guarantee that there is a solution, how do we construct the solution? There are two fundamental pieces to this.

1. *Homogeneous solution.* Ignore all data for now and let $b(x) = 0$.
2. *General solution.* Find the general solution of the ODE.

**Definition 7.1** (Linearly independent function)**.** A set of functions $\{f_n(x)\}$ is a linearly independent set if

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \tag{7.3}$$

for all $x$ only when $c_i = 0$.

**Proposition 7.1.** *An $n^{th}$ order linear homogeneous ODE has exactly $n$ linearly independent solutions. We call this the fundamental set of solutions.*

**Theorem 7.2** (Superposition Principle)**.** *If $y_1(x), \ldots, y_k(x)$ solves a linear homogeneous ODE, then so does*

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x). \tag{7.4}$$

*Meaning, scaling our solution by some constant is still a valid solution.*

**Proposition 7.2.** *If $\{y_1(x), \ldots, y_n(x)\}$ is a fundamental set of solutions, then*

$$y = c_1 y_1(x) + \cdots + c_n y_n(x) \tag{7.5}$$

*is the general solution. By making specific choices for the constant $c_i$ in (7.5), then we can create all possible general solutions.*

**Definition 7.2** (WronSkian)**.** The WronSkian is

$$W = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{pmatrix} \tag{7.6}$$

**Theorem 7.3.** *A set of functions $\{f_1(x), \ldots, f_n(x)\}$ is linearly independent iff. the WornSkian does not equal to 0.*

**Example 7.1.** Let $P = \{\underbrace{\sin(2t)}_{=x_1}, \underbrace{\cos(2t)}_{=x_2}\}$ and confirm that $P$ is a fundamental set of solutions for the equation

$$x''(t) + 4x(t) = 0 \tag{7.7}$$

and find the solutions with the initial data $x\left(\frac{\pi}{6}\right) = 0$ and $x'\left(\frac{\pi}{6}\right) = 4$.

If we take the derivative of the first function, $\sin(2t)$, we have $2\cos(2t)$. If we take the second derivative, we have $-4\sin(2t)$. If we do the same for the second equation in the set, we see that the second derivative is $-4\cos(2t)$. If we then solve

$$\begin{cases} f_1(x) = x_1'' + 4x_1 = -4\sin(2t) + 4\sin(2t) = 0 \\ f_2(x) = x_2'' + 4x_2 = -4\cos(2t) + 4\cos(2t) = 0 \end{cases} \tag{7.8}$$

we can see that $x_1, x_2$ are solutions of the ODE. We then compute the *WronSkian determinant*

$$\begin{aligned} W &= \det \begin{pmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{pmatrix} = \det \begin{pmatrix} \sin(2t) & \cos(2t) \\ 2\cos(2t) & -2\sin(2t) \end{pmatrix} && (7.9) \\ &= -2\sin^2(2t) - 2\cos^2(2t) && (7.10) \\ &= 2 \neq 0. && (7.11) \end{aligned}$$

Therefore, $P$ is a fundamental set of solutions because $x_1, x_2$ solve the ODE and are linearly independent. So,

$$x = c_1 x_1(x) + c_2 x_2(x) \tag{7.12}$$

is the general solution of $x'' + 4x = 0$ (contains all possible solutions) for some $c_i \in \mathbb{R}$. If we then substitute our provided initial data, we have

$$x' = 2c_1\cos(2t) - 2c_2\sin(2t). \tag{7.13}$$

We can then construct the system of matrices via doing some algebra:

$$\begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}. \tag{7.14}$$

Then by solving for and finding that $\begin{pmatrix} c_1 & c_2 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}^\mathsf{T}$, we finally have

$$x(t) = x_1 - \sqrt{3}x_2 = \sin(2t) - \sqrt{3}\cos(2t). \tag{7.15}$$

**Theorem 7.4.** *If $y_p$ is any solution of the inhomogeneous ODE and $y_h$ is the general solution of the homogeneous ODE, then $y_h + y_p$ is the general solution of the inhomogeneous ODE.*

**Example 7.2.** Consider the equations

$$\begin{cases} y'' - y = x^2 \\ y_h = c_1 e^x + c_2 \sinh(x) \end{cases} . \tag{7.16}$$

Now if we have some $y_p = -x^2 - 2$, we have the first and second derivatives to be

$$y_p' = -2x, y_p'' = -2 \tag{7.17}$$

which we can see are continuous for all $\mathbb{R}$ since they are both polynomials. We then know that

$$y = c_1 e^x + c_2 \sinh(x) - x^2 - 2 \tag{7.18}$$

which is the general solution of $y'' - y = x^2$ where $c_i \in \mathbb{R}$.

**Quick Summary.** We have a linear ODE in the form

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = b \tag{7.19}$$

and are able to do the following:

1. Find/verify $n$ solutions of the homogeneous problem.
2. Check linear independence.
3. Use this fundamental set of solutions to write $y_h$, the general solution of the homogeneous problem.
4. Find/check $y_p$, the particular solution to the inhomogeneous ODE.
5. Write $y_h + y_p$, the general solution of the inhomogeneous problem.
6. Fit data.

## 8. Constant Coefficient Linear ODE

**Theorem 8.1.** *Every ordinary differential equation in the form*

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \tag{8.1}$$

*where $a_i \in \mathbb{R}$ has at least one exponential solution in the form $e^{rx}$.*

**Example 8.1.** Take the equations

$$y''' = 2y'' - y' - 2y = 0, y = e^{rx}. \tag{8.2}$$

and through some substitution and factoring, arrive at the set

$$r = \{-2, -1, 1\}. \tag{8.3}$$

We then compute the WronSkian,

$$W = \det \begin{pmatrix} e^{-2x} & e^{-x} & e^{x} \\ -2e^{-2x} & -e^{-x} & e^{x} \\ 4e^{-2x} & e^{-x} & e^{x} \end{pmatrix} = \det \begin{pmatrix} e^{-2x} & e^{-x} & e^{x} \\ -3e^{-2x} & -2e^{-x} & 0 \\ 3e^{-2x} & 0 & 0 \end{pmatrix} \tag{8.4}$$

$$= e^{x} \det \begin{pmatrix} -3e^{-2x} & -2e^{-x} \\ 3e^{-2x} & 0 \end{pmatrix} \tag{8.5}$$

$$= 6e^{-2x} \neq 0 \forall x \tag{8.6}$$

and show linear independence, meaning $\{y_1, y_2, y_3\}$ is a fundamental set of solutions. We have now confirmed that

$$y'' + 4y' + 4y = 0 \tag{8.7}$$

has at least one exponential solution in the form $y = e^{rx}$. We can then solve for these roots via

$$e^{rx}\left(r^2 + 4r + 4\right) = 0 \tag{8.8}$$

$$r^2 + 4r + 4 = 0 \tag{8.9}$$

$$(r+2)(r+2) = 0 \tag{8.10}$$

$$\tag{8.11}$$

which yields $r = -2, -2$. Umm... I suppose we are unable to continue, we need a stronger theorem to work off of. The farthest we can get is figuring out

$$y_1 = e^{-2x}, y_2 = e^{-2x} \tag{8.12}$$

which is *not* an independent set.

**Theorem 8.2.** *Every linear, homogeneous, constant coefficient ODE has solutions of the form*

$$\sum_{i=1}^{\cdots} \rho_i(x) e^{r_i x} \tag{8.13}$$

*which are the only solutions. The $\rho_i(x)$ polynomial with degree no greater than $m_i - 1$ where $m_i$ is the multiplicity of $r_i$.*

Now, if we were to continue Example 8.1, we can see that the multiplicity $m = 2$ and we will have

$$y_1 = \underbrace{1}_{\text{Degree 0 polynomial}} \cdot e^{-2x} \tag{8.14}$$

and $y_2$ be a solution in the form

$$y_2 = (ax + b)e^{-2x}. \tag{8.15}$$

We can then do something for which I was not paying attention, but the steps are as follows:

$$y_2' = ae^{-2x} + -2(ax + b)e^{-2x} \tag{8.16}$$

$$= (-2ax + a - 2b)e^{-2x} \tag{8.17}$$

and similarly

$$y_2'' = (4ax - 4a + 4b)e^{-2x}. \tag{8.18}$$

As we know, an exponential times a polynomial is still continuous, therefore

$$(4ax - 4a + 4b)e^{-2x} + 4(-2ax + a - 2b)e^{-x} + 4(ax + b)e^{-2x} = (0x + 0)e^{-2x} = 0. \tag{8.19}$$

We can then find the general solution of the ODE to be

$$y = C_1e^{-2x} + C_2xe^{-2x} = (C_1 + C_2x)e^{-2x}. \tag{8.20}$$

**Theorem 8.3.** *The solutions of a constant coefficient linear differential equation is always in the form of*

$$e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^ne^{rx}. \tag{8.21}$$

*Remark* 8.1. So basically, if you do things Dr. Goulet's way, we will always get a fundamental set of solutions. Also I absolutely hate this topic so... terrible notes I guess.

**Example 8.2.** If we have some equation

$$y'' + 4y = 0,\ r^2 + 4 = 0 \implies r = \pm 2i. \tag{8.22}$$

We can write our set of solutions in the form described in Theorem 8.3, we have

$$y_1 = e^{2ix},\ y_2 = e^{-2ix} \text{ and } e^{i\theta} = \cos(\theta) + i\sin(\theta),\ e^{-i\theta} = \cos(\theta) - i\sin(\theta). \tag{8.23}$$

Now if we recall the superposition principle, any linear combination is also a solution. So, if we scale our solutions by 1/2, we have

$$\frac{1}{2}e^{2ix} + \frac{1}{2}e^{-2ix} = \cos(2x),\ \frac{1}{2i}e^{2ix} - \frac{1}{2i}e^{-2ix} = \sin(2x). \tag{8.24}$$

We can then see that $\{\sin(2x), \cos(2x)\}$ is a set of solutions. Following this, we can now compute the WronSkian determinant via

$$\mathcal{W} = \det\begin{pmatrix} \sin(2x) & \cos(2x) \\ 2\cos(2x) & -2\sin(2x) \end{pmatrix} = -4 \neq 0 \tag{8.25}$$

which tells us that our solutions are linearly independent. This gives us our general solution,

$$C_1e^{2ix} + C_2e^{-2ix}. \tag{8.26}$$

However, $C_3\sin(2x) + C_4\cos(2x)$ is also a general solution. Let $r = \alpha \pm i\beta$ and rewrite as

$$y_1 = e^{(\alpha+i\beta)},\ y_2 = e^{\alpha - i\beta}. \tag{8.27}$$

From there, we work out that

$$r = \pm\sqrt{2},\ r = 0 \pm i\sqrt{2} \tag{8.28}$$

which tells us that $r = \sqrt{2}e^{\pi ik/2}$ for some $k \in \mathbb{N}$. We can now write our general solution as

$$y = C_1e^{\sqrt{2}x} + C_2e^{-\sqrt{2}x} + C_2\cos(\sqrt{2}x) + C_4\sin(\sqrt{2}x). \tag{8.29}$$

After some algebra, we can see that $r = -1 \pm 1 \cdot i,\ r = -(\pm)i$, giving us the final solution

$$y = (C_1 + C_2x)e^{-x}\cos(x) + (C_3 + C_4x)e^{-x}\sin(x). \tag{8.30}$$

**Example 8.3** (Using Hooke's Law)**.** Say we have some block on a spring connected to a ceiling where $y(t)$ is the position of the center of mass of the block. For this example, we will consider the following:

1. Gravity
2. Dampening (from the spring)
3. Hookean Springs (an ideal spring)

From classical physics, recall a crude definition of *Hooke's Law*: The springs return force is proportional to how much it has stretched from its natural length.

Also from classical physics, recall one of Newton's Laws which states that $\sum \boldsymbol{F} = m\boldsymbol{a}$ and and express our system as

$$m \cdot y''(t) = -mg + k(-y(t) - L). \tag{8.31}$$

Unfortunately, as we continue, we will end up with $f(y'(t))$ which we can approximate using $(y')^2$ which makes our equation non-linear and therefore making it not possible for us to solve at this time. But whatever, if we do try and keep going, we have

$$my''(t) + \alpha y'(t) + ky(t) = -mg - kL. \tag{8.32}$$

Which gives us a linear, inhomogeneous, constant coefficient, second order ODE. One piece of information we can figure out however is an equation for the block's equilibrium, when $y''(t) = y'(t) = 0$ which yields

$$y_{\text{eq}} = \frac{-mg}{k - L}. \tag{8.33}$$

Let $U$ be the displacement from equilibrium and rewrite as

$$y(t) = U + y_{\text{eq}}. \tag{8.34}$$

We have a theorem that tells us that there is at least one solution in the form $U = e^{rt}$. Following this, we can rephrase our equation,

$$(mr^2 + \alpha r + k)e^{rt} = 0 \implies mr^2 + \alpha r + k = 0. \tag{8.35}$$

We can then solve for $r$ using the quadratic formula via

$$r = -\frac{\alpha \pm \sqrt{\alpha^2 - 4mk}}{2m} = -\frac{\alpha}{2m} \pm \sqrt{\left(\frac{\alpha}{2m}\right)^2 - \frac{k}{m}} = -\phi \pm \sqrt{\phi^2 - \gamma}. \tag{8.36}$$

This gives us three different cases:

1. $\phi^2 > \gamma$ gives us two distinct real roots.
2. $\phi^2 = \gamma$ gives us two real repeated roots.
3. $\phi^2 < \gamma$ gives us two complex solutions ($\phi = 0 \implies$ purely imaginary).

If we consider case one, we have

$$U = C_1 e^{\left(-\phi + \sqrt{\phi^2 - \gamma}\right)t} + C_2 e^{\left(-\phi - \sqrt{\phi^2 - \gamma}\right)t}. \tag{8.37}$$

This is becoming very very terrible and I will not continue. This problem can be better answered in MA222, not this course. Fine, case three gives us something like

$$U = C_1 e^{-\phi t} \cos\left(t\sqrt{\gamma - \phi^2}\right) + C_2 e^{-\phi t} \sin\left(t\sqrt{\gamma - \phi^2}\right) \tag{8.38}$$

and so on... Once again, this problem is quite disgusting so we're not going to do it.

Moving on, given some constant coefficient and linear ODE (in-homogeneous) in the form

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x) \tag{8.39}$$

we can find (a) the homogenous solution, $y_h$ and (b) the particular solution $y_p$. The left side of (8.39) are the terms that describe the laws governing our system. When we set those terms equal to 0, we do not apply any external inputs. When we set those terms equal to some $b(x)$, we are providing external inputs (also called stimulus).

*Remark* 8.2. Typically, the system responds in a way that is similar to the inputs. A quick toy example being if I have some rigid body and apply a vibration of 50 Hz, I would expect the rigid body to respond by vibrating at 50 Hz.

So, we now have the understanding to say

$$y = y_h + y_p \tag{8.40}$$

where $y_h$ is the natural behavior of the system and $y_p$ is the response to inputs. Which makes sense, the system will do whatever it naturally does in addition to what is done to it externally.

We can also use some tricks and guesses to help us solve these types of problems. If $b$ contains sines, cosines, exponentials, and/or polynomials, then look for a $y_p$ of the same form. This is a pretty good first guess.

**Example 8.4.** Consider the equation

$$y'' + 4y' + 2y = 2x^2 \tag{8.41}$$

and are given the homogeneous solution,

$$y_h = C_1 e^{(-2-\sqrt{6})x} + C_2 e^{(-2+\sqrt{6})x}. \tag{8.42}$$

Since $b(x) = 2x^2$, we are going to *guess* that that our particular solution is also a second degree polynomial in the form

$$y_p = ax^2 + bx + c. \tag{8.43}$$

We can then continue with this guess and take the following derivatives:

$$y_p' = 2ax + b \tag{8.44}$$

$$y_p'' = 2a. \tag{8.45}$$

Since $y_p, y_p', y_p''$ are polynomials, they are continuous in $\mathbb{R}$. We then substitute

$$2x^2 = 2a + 4(2ax + b) - 2(ax^2 + bx + c) \tag{8.46}$$

$$= \underbrace{(-2a)}_{=2} x^2 + \underbrace{(8a - 2b)}_{=0} x + \underbrace{(2a + 4b - 2c)}_{=0} \tag{8.47}$$

which gives us $a = 1, b = -4, c = -9$ and allows us to create our particular solution

$$y_p = -x^2 - 4x - 9 \tag{8.48}$$

We can then construct the general solution as we would normally.

**Example 8.5.** Consider the equation

$$x'' + x' + x = 2\sin(3t). \tag{8.49}$$

We are also given the homogeneous solution,

$$y_h = C_1 e^{t/2} \cos\left(\frac{\sqrt{3}t}{2}\right) + C_2 e^{t/2} \sin\left(\frac{\sqrt{3}t}{2}\right). \tag{8.50}$$

So, we could guess that our particular solution is in the form

$$y_p = A\sin(3t) + B\cos(3t). \tag{8.51}$$

We continue by taking the first and second derivative of $y_p$,

$$y_p' = 3A\cos(3t) - 3B\sin(3t), \tag{8.52}$$

$$y_p'' = -9A\sin(3t) - 9B\cos(3t). \tag{8.53}$$

Next, we then substitute these derivatives of $y_p$ back into (8.49) via

$$\underbrace{(-8A+3B)}_{=2}\sin(3t)+\underbrace{(-8B-3A)}_{=0}\cos(3t)=2\sin(3t) \tag{8.54}$$

which yields the system

$$\begin{cases} B=\frac{-3}{8}A \\ -8A-\frac{9}{8}A=2 \end{cases} \implies A=-\frac{16}{73},\ B=\frac{6}{73}. \tag{8.55}$$

Since we know $x=x_h+x_p$, we can construct our solution as

$$x=C_1e^{t/2}\cos\left(\frac{\sqrt{3}t}{2}\right)+C_2e^{t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)-\frac{16}{73}\sin(3t)+\frac{6}{73}\cos(3t). \tag{8.56}$$

Since this topic can be pretty tough, we will go ahead and continue with another example.

**Example 8.6.** Consider the equation

$$y''+4y+\sin(x). \tag{8.57}$$

with a homogeneous solution

$$y_h=C_1\sin(2x)+C_2\cos(2x). \tag{8.58}$$

Like before, we can assume our particular solution will be in the form

$$y_p=A\sin(x)+B\cos(x). \tag{8.59}$$

However this $y_p$ is the wrong guess for this system. We need a system in which the amplitudes of our waves increase, which our previous system does not support. Our next guess is going to be

$$y_p=(A+Cx)\sin(x)+(B+Dx)\cos(x). \tag{8.60}$$

Fun fact: We are looking for something with *resonance*, when some external input aligns with the behavior of the system. While our new guess is better, we are going to tune it further. Since we won't find $A,B$ anyway and they are already included in $y_h$ we can omit the coefficients $A,B$ since we plan to write $y_h+y_p$ anyway. Our final guess of $y_p$ is given by

$$y_p=Cx\sin(x)+Dx\cos(x) \tag{8.61}$$

with its second derivative being

$$y_p''=C\big[0+2\cos(x)-x\sin(x)\big]+D\big[0-2\sin(x)-x\cos(x)\big]. \tag{8.62}$$

We can now try to solve for our system as much as possible and start simplifying

$$\begin{cases} \sin(x): & -2D=1 \implies D=-1/2 \\ \cos(x): & 2C=0 \implies C=0 \\ x\sin(x): & -C+C=0 \\ x\cos(x): & -D+D=0 \end{cases} \tag{8.63}$$

giving us the general solution

$$y=C_1\sin(x)+C_2\cos(x)-\frac{1}{2}x\cos(x). \tag{8.64}$$

*Remark* 8.3. You cannot form a particular solution before finding the homogeneous solution. We can make a good guess for it, but we cannot be sure without knowing the resonant behavior.

Making guesses and checking is good when the guesses are relatively small, but we need a more defined and elegant method for more complex systems.

Say we have some system

$$mx'' + \alpha x' + kx = \underbrace{f_0 \sin(\omega t)}_{\text{periodic forcing}} . \tag{8.65}$$

When $\alpha = 0$ (undamped system), we have the homogeneous solution

$$x_h = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right). \tag{8.66}$$

From this, we can make the guess for are particular solution as

$$x_p = \begin{cases} A\sin(\omega t) + B\cos(\omega t), & \omega \neq \sqrt{k/m} \\ Ct\sin\left(\sqrt{\frac{k}{m}}t\right) + Dt\cos\left(\sqrt{\frac{k}{m}}t\right), & \omega = \sqrt{k/m} \end{cases}. \tag{8.67}$$

So when $\omega \neq \sqrt{k/m}$, we have

$$x_p = \frac{f_0}{k - \omega^2 m}\sin(\omega t) \tag{8.68}$$

as $\omega \to \sqrt{k/m}$. Therefore, the amplitude of $x_p$ approaches $\infty$. So, as long as $w \neq \sqrt{k/m}$, (8.68) is our particular solution.

**Summary of Undetermined Coefficients.** We are given differential equations in the form

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = b(x) \tag{8.69}$$

where $a_i$ are constants. If we have a homogeneous solution in the form we will call $(\star)$,

$$y_h = \sum P_i(x)e^{\alpha_i x}\cos(\beta_i x) + Q_i(x)e^{\alpha_i x}\sin(\beta_i x) \tag{8.70}$$

we can solve it via the following steps:

1. Find $\frac{1}{h}$ with form $(\star)$.
2. Look at $b(x)$ and if it is in the form $(\star)$, use the methods we have learned in this section. If not, we must use something else which we have not yet learned (just look at the next section).

Us not being able to continue if we do not have a $y_h$ described in (8.70) motivates the next section, the concept of *variation of parameters*.

# 9. VARIATION OF PARAMETERS

We have solved very basic variation of parameters problems in previous classes, such as Calculus I or II but we need to apply this concept to higher order ODEs. Normally, we have a differential equation written as

$$y' + p(x)y = q(x) \tag{9.1}$$

but if we are given a second order ODE, it will be in the form of

$$y'' + a_1(x)y' + a_0(x)y = b(x). \tag{9.2}$$

Since we have the homogeneous solution to be

$$y_h = C_1y_1(x) + C_2y_2(x), \tag{9.3}$$

we can write our solution to the differential equation, $y$, as

$$y = U_1(x)y_1(x) + U_2(x)y_2(x). \tag{9.4}$$

We can continue by plugging (9.4) into (9.2) to get

$$y'' = \Big[U_1'(x)y_1(x) + U_2'(x)y_2(x)\Big] + \Big[U_1(x)y_1'(x) + U_2(x)y_2'(x)\Big]. \tag{9.5}$$

Since we have two unknown coefficients $U_1, U_2$, we will need two equations. Our first choice is

$$U_1'(x)y_1(x) + U_2'(x)y_2(x) = 0. \tag{9.6}$$

We can then compute the second derivative of $y$ which gives (we are going to stop writing our functions as functions of $x$ because it would take decades)

$$y'' = \Big[U_1'y_1' + U_2'y_2'\Big] + \Big[U_1y_1'' + U_2y_2''\Big]. \tag{9.7}$$

Then, we can plug $y''$ into our original ODE to get

$$U_1\Big[\underbrace{y_1'' + a_1y_1' + a_0y_1}_{=0}\Big] + U_2\Big[\underbrace{y_2'' + a_1y_2' + a_0y_2}_{=0}\Big] + U_1'y_1' + U_2'y_2' = b(x). \tag{9.8}$$

If we have the two equations

$$\begin{cases} U_1'y_1 + U_2'y_2 = 0 \\ U_1'y_1' + U_2'y_2' = b \end{cases}. \tag{9.9}$$

As per usual, we need to determine if this system has a unique solution, meaning, we must compute the WronSkian via

$$\mathcal{W} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} U_1' \\ U_2' \end{pmatrix} = \begin{pmatrix} U \\ b \end{pmatrix} \tag{9.10}$$

and since $\{y_1, y_2\}$ are linearly independent because $c_1y_1 + c_2y_2 = y_h$, $\det(\mathcal{W}) \neq 0$ which means that there exists a $\mathcal{W}^{-1}$, implying the system has a unique solution. However, I will point out that I have absolutely no clue what is going on.

*Remark* 9.1. Luckily, this method does not increase in difficulty. You can have a differential equation with however many terms and it would be basically the same amount of work.

If you recall, we once looked at a heat conduction problem within a sphere whose system is defined via the differential equation

$$U'' + \frac{2}{r}U' = -\frac{s}{k} \tag{9.11}$$

with the homogeneous solution being

$$U_h = C_1 + C_2\frac{1}{r}. \tag{9.12}$$

As per usual, we are able to replace $C_i$, our coefficients, with functions of $r$ named $f$,

$$U = f_1(r) + f_2(r)\frac{1}{r} \tag{9.13}$$

which we can take the first derivative of to get

$$U' = f_1'(r) + f_2'(r)\frac{1}{r} - f_2(r)\frac{1}{r^2}. \tag{9.14}$$

Then, we can substitute these back into (9.11) to get

$$U'' = -f_2'(x)\frac{1}{r^2} + 2f_2(r)\frac{1}{r^3} - f_2'(r)\frac{1}{r^2} + 2f_2(r)\frac{1}{r^3} + \frac{2}{r}\left(-f_2(x)\frac{1}{r^2}\right) = -\frac{s}{k} \tag{9.15}$$

which upon simplifying, gives us

$$-f_2'(r)\frac{1}{r^2} = -\frac{s}{k} \implies \begin{cases} f_2'(r) = \frac{s}{k}r^2 \\ f_1'(r) + f_2'(r)\frac{1}{r} = 0 \end{cases}. \tag{9.16}$$

Therefore,

$$f_1 = -\frac{s}{2k}r^2 + C_1,\ f_2 = \frac{s}{3k}r^3 + C_2 \tag{9.17}$$

which gives us the general solution of

$$U = f_1 + \frac{1}{r}f_2 = -\frac{2}{2k}r^2 + \frac{s}{2k}r^3 + U_h. \tag{9.18}$$

**Example 9.1** (Euler's Differential Equation)**.** Euler's name is in front of everything but there is a type of ODE that is indeed called that. An example is

$$t^2 + y'' - ty' + y = t \tag{9.19}$$

with the homogeneous solution of

$$y_h = C_1 + C_2 + \ln(t),\ t > 0 \tag{9.20}$$

which we can rewrite as

$$y_h = U_1(t)t + U_2(t) + \ln(t). \tag{9.21}$$

We then take the first and second derivatives via

$$y' = \underbrace{U_1't + U_2' + \ln(t)}_{=0} + U_1 + U_2(\ln(t) + 1) \tag{9.22}$$

$$y'' = U_1' + U_2'(\ln(t) + 1) + U_2\frac{1}{t}. \tag{9.23}$$

Once again, we then substitute back into the original equation to get

$$t^2\left(U_1' + U_2'(\ln(t) + 1) + U_2\frac{1}{t}\right) + (U_1 + U_2(\ln(t) + 1)) + U_1t + U_2t\ln(t) = t. \tag{9.24}$$

Eventually, this will give us the matrix-vector system

$$\begin{pmatrix} 1 & \ln(t) + 1 \\ 1 & \ln(t) \end{pmatrix}\begin{pmatrix} U_1' \\ U_2' \end{pmatrix} = \begin{pmatrix} 1/t \\ 0 \end{pmatrix} \implies \left(\begin{array}{cc|c} 0 & 1 & 1/t \\ 1 & \ln(t) & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 0 & 1 & 1/t \\ 1 & 0 & -\ln(t)/t \end{array}\right) \tag{9.25}$$

which tells us that

$$U_1 = -\int \frac{\ln(t)}{t}\mathrm{d}t = -\frac{1}{2}(\ln(t))^2 + C_1,\ U_2 = \ln(t) + C_2. \tag{9.26}$$

*Remark* 9.2. Dr. Goulet got destroyed in chess I guess, very important comment for my notes. He spent his time watching videos about chess rather than actually playing chess, so I suppose this is an important reminder to us all.

Anyway, continuing with the example, we end up getting

$$y = -\frac{1}{2}t(\ln(t))^2 + t(\ln(t))^2 + y_h. \tag{9.27}$$

Let's go back and revisit the most basic form of an ODE,

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = b(x) \tag{9.28}$$

and if we have a homogeneous solution as

$$y_h = \sum_{i=1}^{n} C_1 y_i(x), \quad y = \sum_{i=i}^{n} U_i(x)y_i(x) \tag{9.29}$$

which we can take the $n$-th derivative of via

$$y' = \underbrace{\sum_{i=1}^{n} U_i'(x)y_i(x)}_{=0} + \sum_{i=1}^{n} U_i(x)y_i'(x) \tag{9.30}$$

$$y'' = \underbrace{\sum_{i=1}^{n} U_i'(x)y_i'(x)}_{=0} + \sum_{i=1}^{n} U_i(x) + y_i''(x) \tag{9.31}$$

$$\vdots \tag{9.32}$$

$$y^{(n)} = \underbrace{\sum_{i=1}^{n} (U_i'(x)y_i^{(n-1)}(x)}_{=0} + \sum_{i=1}^{n} U_i(x)y^{(n-1)}(x). \tag{9.33}$$

This gives us something extremely silly that we can actually use to bring back our best friend, the WronSkian determinant which we find via

$$\mathcal{W} = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} U_1' \\ U_2' \\ \vdots \\ U_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{pmatrix}. \tag{9.34}$$

Alas, I have absolutely no clue what I am doing right now but I do know that from the equation $y^{(4)} - y = b(x)$, we will have the roots

$$(r-1)(r+1)(r-i)(r+i) = 0. \tag{9.35}$$

Then, we can write our homogeneous solution as

$$y_h = C_1e^x + C_2e^{-x} + C - 3\sin(x) + C_4\cos(x). \tag{9.36}$$

After some terrible, terrible work, we have $y$ defined via

$$y = U_1(x)\sin(x) + U_2(x)\cos(x) + U_3(x)\sinh(x) + U_4(x)\cosh(x) \tag{9.37}$$

which we can shove into an evil matrix which I am not even going to do. If you wanted me to do so please cry yourself to sleep for even the great Dave Goulet had two minutes left in class and basically plugged into a formula.

Practical resonance is only possible if $2mk - \alpha^2 > 0$. Anyway, here is an example with some practical resonance.

**Example 9.2.** Consider the system described by the differential equation

$$x'' + \alpha x' + x = 5\sin(\omega t) \tag{9.38}$$

exhibits *steady state oscillations* with *maximum amplitude* when $\omega = \frac{1}{2}$. We are going to find $\alpha$. First, we need to find the homogeneous solution of the differential equation via the characteristic polynomial

$$r^2 + \alpha r + r = 0 \tag{9.39}$$

which we can solve using the quadratic equation via

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4(1)(1)}}{2(1)} = \frac{-\alpha \pm \sqrt{\alpha^2 - 4}}{2}, \tag{9.40}$$

giving us a homogeneous solution in the form

$$x_h = C_1 e^{\frac{-\alpha + \sqrt{\alpha^2-4}}{2}t} + C_2 e^{\frac{-\alpha - \sqrt{\alpha^2-4}}{2}t},\ C_i \in \mathbb{R}. \tag{9.41}$$

We can then find the form of the particular solution to be

$$x_p = a\sin(\omega t) + b\cos(\omega t) \tag{9.42}$$

where $a, b \in \mathbb{R}$. We then take the first and second derivatives of $x_p$ via

$$\begin{aligned}
x_p' &= a\omega\cos(\omega t) - b\omega\sin(\omega t), && (9.43)\\
x_p'' &= a\big(\omega \cdot -\omega\sin(\omega t) + \cos(\omega t)\cdot 1\big) - b\big(\omega\cdot\omega\cos(\omega t) + \sin(\omega t)\cdot 1\big) && (9.44)\\
&= a\big(-\omega^2\sin(\omega t) + \cos(\omega t)\big) - b\big(\omega^2\cos(\omega t) + \sin(\omega t)\big) && (9.45)\\
&= -a\omega^2\sin(\omega t) + \cos(\omega t) - b\omega^2\cos(\omega t) + \sin(\omega t) && (9.46)\\
&= \cos(\omega t)(a - b\omega^2) + \sin(\omega t)(-a\omega^2 - b). && (9.47)
\end{aligned}$$

We then plug these monstrous derivatives back into (9.38) to get

$$\cos(\omega t)(a - b\omega^2) + \sin(\omega t)(-a\omega^2 - b) + \alpha\left(a\omega\cos(\omega t) - b\omega\sin(\omega t)\right) = a\sin(\omega t) + b\cos(\omega t). \tag{9.48}$$

which we can simplify to

$$\begin{aligned}
a\sin(\omega t) + b\cos(\omega t) &= \cos(\omega t)(\underbrace{a - b\omega^2 + a\alpha w}_{=0}) + \sin(\omega t)(\underbrace{-a\omega^2 - b - \alpha b\omega}_{=5}) && (9.49)\\
&= \cos(\omega t)(0) + \sin(\omega t)(5) && (9.50)\\
&= 5\sin(\omega t) && (9.51)
\end{aligned}$$

giving us the equation

$$5\sin(\omega t) = a\sin(\omega t) + b\cos(\omega t) \tag{9.52}$$

which we can rewrite as the final particular solution

$$x_p = 5\sin(\omega t). \tag{9.53}$$

Although, we still need to solve for $a$ and $b$. From (9.52), we can match coefficients to determine that

$$a = 5,\ b = 0. \tag{9.54}$$

Okay something is terribly wrong and this problem is disgusting to do by hand. Never mind.

# 10. Second Order Numerical Approximations

We should never have to solve higher order ODEs since we can turn them into a set of lower order ODEs. For instance, consider the equation

$$x'' + 2x' + 3x = t \tag{10.1}$$

in which we let $y = x'$ and rewrite $y' = t = 3x = 2y$ which we use to solve the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}. \tag{10.2}$$

Now consider the differential equation

$$\theta'' + \frac{g}{\ell} \sin(\theta) = 0 \tag{10.3}$$

where we let $\theta' = \omega$ and rewrite $\omega' = -\frac{g}{\ell} \sin(\theta)$. We then define $x^{(n)}$ via

$$x^{(n)} = f\left(x, x', x'', \dots, x^{(n-1)}\right). \tag{10.4}$$

Our problems suddenly become much easier to solve when we let $x = y'$ and solve whatever equations we need to. The point is, we don't really need to learn a thousand methods to solve higher order differential equations. Actually, if you recall from the start of the course, we will ever only learn to solve an extremely small subset of differential equations in our lifetimes.

In section 3, we discussed *Euler's method* in which we have some first order differential equation

$$x'(t) = f(x, t) \tag{10.5}$$

and we can approximate $x_{n+1}$ via

$$x_{n+1} = x_n + hf(x_n, t_n), \ h = \Delta t. \tag{10.6}$$

Of course, we can also continue with Heun's method, midpoint method, RK4, etc. The point is, we can use whatever numerical approximation method we want and similarly, we can extend this idea to second order ODEs. Despite the fact we know how to solve this by hand, consider the following example.

**Example 10.1** (Euler's Method)**.** Consider the differential equation and initial data given by

$$x'' + 2x' + x = \cos(t); \ x(0) = 0, \ x'(0) = 1 \tag{10.7}$$

We can let $x' = y$ and rewrite

$$y' = \cos(t) - x - 2y; \ x(0) = 0, \ y(0) = 1. \tag{10.8}$$

Our goal will be to estimate the value of $x\left(\frac{1}{2}\right)$ using two steps of Euler's method. Where $x_0 = 0$ and $y_0 = 1$, we perform the following steps:

$$\begin{align} x_1 &= x_0 + hf(x_0, y_0, t_0) \tag{10.9} \\ &= 0 + \frac{1}{4}(1) = \frac{1}{4}, \tag{10.10} \\ y_1 &= y_0 + hg(x_0, y_0, t_0) \tag{10.11} \\ &= 1 + \frac{1}{4}(-1) = \frac{3}{4}, \tag{10.12} \\ t_1 &= t_0 + h \tag{10.13} \\ &= 0 + \frac{1}{4} = \frac{1}{4}. \tag{10.14} \end{align}$$

Then, we can perform another step to get

$$x_2 = x_1 + hf(x_1, y_1, t_1) = \frac{1}{4} + \frac{1}{4}\frac{3}{4} = \frac{7}{16}. \tag{10.15}$$

Therefore, $x\left(\frac{7}{16}\right)$ is approximated by $\frac{7}{16}$ via two steps of Euler's method.

A good use of numerical methods is solving differential equations that describe pendulums. This is because these differential equations do not have general solutions. For instance, we can describe the angle $\theta$ of a pendulum via

$$\theta'' + \frac{g}{\ell}\sin(\theta) = 0. \tag{10.16}$$

This differential equation will be solved using *Maple* but I am not going to solve it, you can figure it out on your own by just using the methods you should have already learned.

Wow, we have officially discussed all topics in the course, this section wasn't as much as I thought it would be, this is basically just normal numerical approximations.